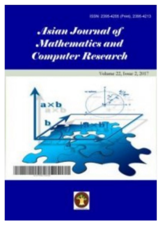
Asian Journal of Mathematics and Computer Research

25(1): 18-37, 2018

ISSN: 2395-4205 (P), ISSN: 2395-4213 (O)



FIXED POINT THEOREMS IN DIGITAL IMAGES AND APPLICATIONS TO FRACTAL IMAGE COMPRESSION

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AUTHORS' CONTRIBUTIONS

This work was carried out in collaboration between both authors. Author VVN designed the study, performed the theoretical analysis, wrote the protocol and wrote the first draft of the manuscript and managed literature searches. Author UPD managed the modifications and corrections of the study and literature searches. Both authors read and approved the final manuscript.

ARTICLE INFORMATION

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***Received:** 10th February 2018*
***Accepted:** 27th April 2018*
***Published:** 7th May 2018*

Original Research Article

ABSTRACT

In this research paper we prove some fixed point theorems for digital images. Ege and Karaca stated and proved Banach contraction principle for digital images. Main objective of the research article is to present another generalization of the well known Banach contraction mapping principle for digital images. We generalize the principle by replacing the contraction condition of Banach by a condition that involves monotone non-decreasing function. In the second result, we use a weakly uniformly strict digital contraction to prove the existence of unique fixed point for digital images. The basic concepts about the digital images are mentioned. We give an important application of our fixed point theorem to compression of digital images. Fractal image compression is one of the popular technique for compressing a digital image. It is based on the self similarity search of the image. But it has a major drawback of computational intensity in encoding a digital image. Computational intensity increases the time of data transmission. In this paper a technique is proposed to bring down the time of data transmission. In an image compression, it is a challenge to either maximize

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the image quality for a stipulated data transmission time or to minimize the data transmission time for a given quality of an image to be transmitted. To achieve this goal, a constant contractive factor in conventional fractal image compression is replaced by the non-linear contractive mapping. This leads to significantly better reconstruction of image in lesser time. Finally we mention some conclusions about our research article.

Key words: Digital image; digital metric space; fixed points; fractals; fractal image compression.

2010 Mathematics Subject Classification: 47H10, 54H25, 94A08, 68U10.

1 Introduction

It has been said that 21st century era is of information and technology. Now a days the traditional vocal communication is taken over by the new technology of visual communication. Some applications inherently demand the visual dimension in their communications. Computer communications, video conferencing, desktop multimedia publishing, video calling, broadcasting, teleconferencing and image transferring are some examples of these applications. The obvious reason behind the digital visual communication is that the digital image representation of the data allows the user to easily manipulate the visual information in an useful way. So we expect some more aspects to be added in the computer that we use today, so that it become compatible for efficient visual communication.

Although it is undoubtedly true that an image is equivalent to thousands of words, it also demands the management of tremendous amount of data, its storage and transmission. Thus the main problem with the digital images is that the large number of bytes are essential to represent them. For example a digital image of size 1920×1200 with 24 bits per pixel requires about 1.46 MB of a computer memory. Transmission of the image using a 19200 bits per second modem takes 5.5 minutes. This much time consumption is off course not affordable for many applications.

However, this problem can be removed. A digital image contains a considerable quantity of redundancy. Still image contain spatial redundancy, because neighboring pixels are correlated. Colored image has spectral redundancy, due to correlation between different color components (Red, Green, Blue). Video contains temporal redundancy, due to the correlation between different frames. Redundancies, when removed from a digital product (image or video), compress its size considerably. The purpose of the image compression is to remove these redundancies and thereby to reduce the size of the image to represent it for the suitable application. Broadly, all the image compression techniques are categorized as lossless and lossy. In lossless image compression, an image is reversible and in lossy image compression techniques it is irreversible. There are many popular image compression techniques. Some of them are

1. Predictive Coding - Different Pulse Code Modulation (DPCM) [1]
2. Transform Coding [2]
3. Vector Quantization [3]
4. Wavelets Coding [4, 5]
5. Model Based Coding [6]
6. Entropy Coding [7]
7. Fractal Image Compression

Fractal image compression falls into the lossy technique category. Fixed point theory plays an important role in image processing and computer graphics. In this research area the main aim is to obtain key results on digital images, that is to say key fixed point theorems to serve a particular purpose. Fixed point theory is a blend of several areas of mathematics like mathematical analysis, topology and functional analysis. There are various applications of fixed point theory in computer science, game theory, engineering and image processing. In recent years, there have been many developments in digital topology. Ege and Karaca [8] constructed Lefschetz fixed point theory for digital images and studied the fixed point properties of digital images. Ege and Karaca [9] gave relative and reduced Lefschetz fixed point theorems for digital images. They also calculated degree of the antipodal map for sphere-like digital images using fixed point properties. Ege and Karaca [10] proved Banach fixed point theorem for digital images and gave an application to image processing. Kumari Jyoti et.al [11] also proved many useful results in digital images and presented some applications.

In the next section, we give some necessary ideas about digital images. In section 3, we present generalization of Banach fixed point theorem for digital images. We also explore the technique of fractal image compression in detail and present an application of our theorem in the last section.

2 Preliminary Notes

Definition 2.1. Let $\mathbb{Z}$ be the set of all integers and let n be a positive integer. Define the set $\mathbb{Z}^n$ as follows:

$$\mathbb{Z}^n = \{(x_1, x_2. \cdots, x_n)/x_i \in \mathbb{Z}, 1 \leq i \leq n\}$$

$\mathbb{Z}^n$ is also called the set of all lattice points in the n dimensional Euclidean space.

Definition 2.2. [12] Consider any two distinct points $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n)$ in $\mathbb{Z}^n$. Let m be a positive integer such that $1 \leq m \leq n$. We say that the two points p and q are k_m-adjacent in $\mathbb{Z}^n$ if there are at most m indices i such that $|p_i - q_i| = 1$ and for all other indices j such that $|p_j - q_j| \neq 1$, we have $p_j = q_j$.

Example 2.3. [10] Consider $\mathbb{Z}$. Two points p and q in $\mathbb{Z}$ are k_1-adjacent if $|p-q| = 1$. Since there are two points that are k_1-adjacent to a given point in $\mathbb{Z}$, this adjacency relation is given by 2-adjacency.

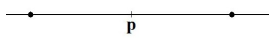

Figure 1: 2-adjacency in Z

Consider $\mathbb{Z}^2$. Then the points $p = (2,3)$ and $q_1 = (3,3)$ are k_1-adjacent in $\mathbb{Z}^2$. The other points in $\mathbb{Z}^2$ that are k_1-adjacent to $p = (2,3)$ are $q_2 = (1,3)$, $q_3 = (2,4)$ and $q_4 = (2,2)$. As there are four points q_1, q_2, q_3, q_4 that are k_1-adjacent to p, we write this adjacency relation as 4-adjacency relation. With this terminology, the points p and q_1 are 4-adjacent, the points p and q_2 are 4-adjacent and so on. Similarly we observe that $q_1, q_2, q_3, q_4, q_5 = (1,2), q_6 = (3,4), q_7 = (3,2)$ and $q_8 = (1,4)$ are k_2-adjacent to $p = (2,3)$. As there are 8 points $q_1, q_2, \cdots, q_8$ that are k_2-adjacent to p, this adjacency relation is written as 8-adjacency relation in $\mathbb{Z}^2$. With this terminology, the points p and the points $q_1, q_2, \cdots q_8$ are 8-adjacent.

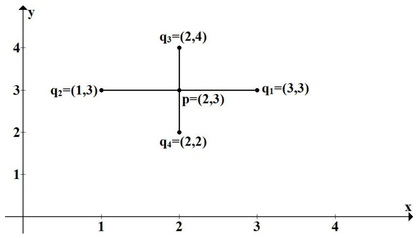

Fig. 2. 4-adjacency in Z^2

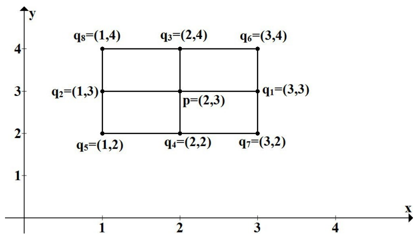

Fig. 3. 8-adjacency in Z^2

In general $k_l$represents the number of points that are kl -adjacent to a given point p.

Example 2.4. Consider $\mathbb{Z}^3$. Two points p and q in $\mathbb{Z}^3$ are k_3-adjacent if they are distinct and differ by at most 1 in each coordinate. Since there are 26 points q that are k_3-adjacent to a given point p, this adjacency relation is written as 26-adjacency relation. Similarly two points p and q in $\mathbb{Z}^3$ are k_2-adjacent if they are distinct and differ by at most 1 in at most two of their coordinates. This adjacency relation is denoted by 18-adjacency relation. Finally, two points in $\mathbb{Z}^3$ are k_1-adjacent if they are distinct and differ by at most 1 in exactly one coordinate. This adjacency relation is denoted by 6-adjacency relation.

Remark 2.5. Some other adjacency relations are discussed in [13].

Definition 2.6. [10] A digital image is an ordered pair (X, κ), where X is a finite subset of $\mathbb{Z}^n$ for some positive integer n and κ is an adjacency relation for the members of X.

The following are the basic notions in digital images.

Definition 2.7. [12] A κ-neighbour of a point $p \in (X, \kappa)$ is a point of X that is κ-adjacent to p, where $\kappa \in \{2, 4, 6, 8, 18, 26\}$ and $X \subset \mathbb{Z}^n$, $n = 1, 2, 3$.

Definition 2.8. [10] The set $N_\kappa(p) = \{q \ / \ q \ is \ \kappa - neighbour \ of \ p\}$ is called the κ-neighbourhood of p.

Definition 2.9. [14] A digital interval is defined by $[a, b]_\mathbb{Z} = \{z \in \mathbb{Z}/a \leq z \leq b\}$, where $a, b \in \mathbb{Z}$ and $a < b$.

Definition 2.10. [13] A digital image (X, κ) is κ-connected if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, x_2, \cdots, x_r\}$ of points of digital image (X, κ) such that $x = x_0$, $y = x_r$ and x_i and x_{i+1} are κ-neighbours, where $i = 0, 1, 2, \cdots, r - 1$.

Definition 2.11. [12] Let $(X, \kappa_0) \subset \mathbb{Z}^m$ and $(Y, \kappa_1) \subset \mathbb{Z}^n$ be a digital images and $T : X \to Y$ be a function. If for every κ_0-connected subset A of X, $T(A)$ is a κ_1-connected subset of Y, then T is said to be (κ_0, κ_1)-continuous.

Definition 2.12. [15] If in the above definition 2.11, T is (κ_0, κ_1)-continuous, bijective and T^{-1} is (κ_1, κ_0)-continuous, then T is called (κ_0, κ_1)-isomorphism. We denote it by $X \cong_{(\kappa_0, \kappa_1)} Y$.

Definition 2.13. A point $x \in (X, d, \kappa)$ is called a fixed point of the mapping $T : X \to X$ if $Tx = x$.

Let (X, κ) be a digital image. We say that the digital image (X, κ) has the fixed point property [10] if every (κ, κ)-continuous map $T : (X, \kappa) \to (X, \kappa)$ has a fixed point.

Definition 2.14. [10] Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital image. Define a function $d : X \times X \to [0, \infty)$ by,

$$d(p, q) = \Big[\sum_{i=1}^{n}(p_i - q_i)^2\Big]^{\frac{1}{2}} \tag{2.1}$$

Then we have the following properties satisfied by d for all $x, y, z \in X$.
1. $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq d(x, z) + d(z, y)$.

The digital image (X, κ) together with the function d is called a digital metric space with κ-adjacency. It is denoted by (X, d, κ).

Definition 2.15. [10] A sequence $\{x_n\}_{n=1}^{\infty}$ of points of a digital metric space (X, d, κ) is a Cauchy sequence if for all $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that for all $m, n > N$, we have $d(x_m, x_n) < \epsilon$.

Definition 2.16. [10] A sequence $\{x_n\}_{n=1}^{\infty}$ of points in a digital metric space (X, d, κ) converge to a limit $L \in X$ if for all $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that for all $n > N$, we have $d(x_n, L) < \epsilon$.

Definition 2.17. [10] A digital metric space (X, d, κ) is said to be a complete digital metric space if every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ of points of (X, d, κ) converge to a point L of (X, d, κ).

Definition 2.18. [10] Let (X, d, κ) be a digital metric space. A function $T : (X, d, \kappa) \to (X, d, \kappa)$ is called right continuous if $\lim\limits_{x \to a^+} Tx = Ta$, where $a \in X$.

Definition 2.19. [10] Let (X, d, κ) be any digital metric space and $T : (X, d, \kappa) \to (X, d, \kappa)$ be a digital self map. If there exists $\lambda \in (0, 1)$ such that for all $x, y \in X$, $d(Tx, Ty) \leq \lambda d(x, y)$, then T is called a digital contraction map. Also the constant λ is called a contractive factor.

Definition 2.20. Let (X, d, κ) be a digital metric space. A self map $T : (X, d, \kappa) \to (X, d, \kappa)$ is called a strict digital contraction if for all $x, y \in X, x \neq y$, $d(Tx, Ty) < d(x, y)$.

Definition 2.21. Let (X, d, κ) be a digital metric space. A self map $T : (X, d, \kappa) \to (X, d, \kappa)$ is called a weakly uniformly strict digital contraction if given $\epsilon > 0$, there exists $\delta > 0$ such that $\epsilon \leq d(x, y) < \epsilon + \delta$ implies $d(Tx, Ty) < \epsilon$ for all $x, y \in X$.

Many of the basic theorems in metric fixed point theory are extended to digital metric space. We mention some of the important ones. Brouwer's fixed point theorem in one dimension for digital images is as follows:

Theorem 2.22. [10] Every $(2, 2)$-continuous function $T : ([0, 1]_{\mathbb{Z}}, d, 2) \to ([0, 1]_{\mathbb{Z}}, d, 2)$ has a fixed point, where $d(x, y) = |x - y|$ for all $x, y \in [0, 1]_{\mathbb{Z}}$.

Brouwer fixed point theorem in two dimensions is stated as follows:

Theorem 2.23. Let $X = \{(0,0),(1,0),(0,1),(1,1)\} \subset \mathbb{Z}^2$ be a digital image with 4-adjacency. Then every (4, 4)-continuous function $T : (X, d, 4) \to (X, d, 4)$ has a fixed point, where $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ for all $x = (x_1, x_2), y = (y_1, y_2) \in X$.

Ege and Karaca [10] formulated and proved the Banach contraction mapping principle for digital images in 2015 as follows:

Theorem 2.24. [10] Let (X, d, κ) be a complete digital metric space. Let $T : (X, d, \kappa) \to (X, d, \kappa)$ be a digital contraction map. Then T has unique fixed point, that is there exists a unique point $z \in X$ such that $Tz = z$.

Ege and Karaca [10] further generalized the above theorem as stated below. We observe that if the function $\psi(t)$ is taken as $\psi(t) = \lambda t$, where $\lambda \in [0, 1)$, we get the Banach contraction mapping principle as stated in theorem 2.24.

Theorem 2.25. [10] Let (X, d, κ) be a complete digital metric space and let $T : (X, d, \kappa) \to (X, d, \kappa)$ be a digital self map. Assume that there exists a right continuous real function $\psi : [0, v] \to [0, v]$, where v is sufficiently large real number such that $\psi(a) < a$ if $a > 0$ and let Tsatisfies $d(Tx_1, Tx_2) \leq \psi(d(x_1, x_2))$ for all $x_1, x_2 \in (X, d, \kappa)$. Then T has a unique fixed point $z \in (X, d, \kappa)$ and the sequence $\{T^n x\}_{n=1}^{\infty}$ converge to z for every $x \in X$.

Recently Kumari Jyoti and Asha Rani[11] presented an application of fixed point theory of digital metric space in image processing. They have proved that expansive mappings on complete digital metric space have a fixed point.

Theorem 2.26. [11] Let $T : (X, d, \kappa) \to (X, d, \kappa)$ be a mapping on a complete digital metric space X. Let Tbe onto and satisfy

$$d(Tx, Ty) \geq \lambda d(x, y)$$

for all $x, y \in X$and $\lambda > 1$. Then T has a fixed point in X.

Remark 2.27. [11] The mapping T in the above theorem can be replaced by a bijective mapping.

The condition on the mapping Tin the above theorem 2.26 is replaced by another suitable condition and the following results are obtained.

Theorem 2.28. [11] Let $(X, d, , \kappa)$ be a complete digital self map which is continuous and onto on X. Let T satisfy the condition

$$d(Tx, Ty) \geq \lambda\mu$$

where $\lambda > 1$, and

$$\mu = \mu(x, y) \in \left\{d(x, y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

then T has a fixed point.

Remark 2.29. [11] It has been proved that μ in the above theorem may be replaced by

$$\mu = \mu(x, y) \in \left\{d(x, y), \frac{d(x,Tx)+d(y,Ty)}{2}, d(x, Ty), d(y, Tx)\right\}$$

3 Main Results

In the following theorem we replace the continuity condition on ψ in the theorem 2.25 by another suitable condition.

Theorem 3.1. Let (X, d, κ) be a complete digital metric space and suppose that $T : (X, d, \kappa) \to (X, d, \kappa)$ satisfies $d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$, where $\psi : [0, \infty) \to [0, \infty)$ is monotone nondecreasing and satisfy $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$. Then T has a unique fixed point in (X, d, κ).

Proof. Let x_0 be an arbitrary but fixed element in (X, d, κ). Define a sequence of iterates $\{x_n\}_{n=1}^{\infty}$ in X by
$x_1 = Tx_0$, $x_2 = Tx_1$, $x_3 = Tx_2$, $\cdots$, $x_n = Tx_{n-1} \cdots$. Note that,

$$\begin{aligned} 0 \leq d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \psi(d(x_n, x_{n-1})) \\ &= \psi(d(Tx_{n-1}, Tx_{n-2})) \\ &\leq \psi(\psi(d(x_{n-1}, x_{n-2}))) \\ &= \psi^2(d(x_{n-1}, x_{n-2})) \end{aligned}$$

Continuing in this way we get

$$0 \leq d(x_{n+1}, x_n) \leq \psi^n(d(x_1, x_0))$$

Thus

$$0 \leq \limsup_{n\to\infty} d(x_{n+1}, x_n) \leq \limsup_{n\to\infty} \psi^n(d(x_1, x_0)) = 0$$

Hence

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$$

We now show that the sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Also note that for any $\epsilon > 0$, $\psi(\epsilon) < \epsilon$. And since

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0,$$

so for $\epsilon > 0$, we can choose n such that $d(x_{n+1}, x_n) \leq \epsilon - \psi(\epsilon)$. Now define the set $S = \{x \in X / d(x, x_n) < \epsilon\}$. Then for any $y \in S$, we have

$$\begin{aligned} d(Ty, x_n) &\leq d(Ty, Tx_n) + d(Tx_n, x_n) \\ &\leq \psi(d(y, x_n)) + d(x_{n+1}, x_n) \\ &\leq \psi(\epsilon) + \epsilon - \psi(\epsilon) \\ &= \epsilon \end{aligned}$$

Thus $Ty \in S$. Hence $T(S) \subset S$. Therefore $d(x_m, x_n) \leq \epsilon$ for all $m \geq n$. Hence the sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since (X, d, κ) is digital complete metric space, there is a limit z of $\{x_n\}_{n=1}^{\infty}$ in (X, d, κ). Now we observe that the function T is (κ, κ)-continuous. If $a \in X$ and $\epsilon > 0$, then let $\delta = \epsilon$. Thus if $d(a, b) < \delta$, we have

$$\begin{aligned} d(Ta, Tb) &\leq \psi(d(a, b)) \\ &< d(a, b) \\ &< \epsilon \end{aligned}$$

Thus T is (κ, κ)-continuous function. From the (κ, κ)-continuity of T we get

$$z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = T\Big[\lim_{n\to\infty} x_{n-1}\Big] = Tz$$

Therefore, T has a fixed point z.

Uniqueness- Assume that $u, v \in X$ are fixed points of T. Then we have

$$d(u, v) = d(Tu, Tv) \leq \psi((d(u, v))) < d(u, v)$$

This imply $d(u, v) = 0$ and hence $u = v$. □

Remark 3.2. If we take $\psi(t) = \lambda t$ in the theorem 3.1, where $\lambda \in (0, 1)$, we get the Banach contraction principle. Thus the theorem 3.1 is a generalization of Banach contraction mapping principle.

In the following theorem we use the weakly uniformly strict digital contraction to prove the existence of unique fixed point theorem for digital images.

Theorem 3.3. Let (X, d, κ) be a complete digital metric space and $T : (X, d, \kappa) \to (X, d, \kappa)$ be a weakly uniformly strict digital contraction mapping. Then T has a unique fixed point z. Moreover, for any $x \in X$, $\lim_{n\to\infty} T^n x = z$.

Proof. We first observe that the weakly uniformly strict digital contraction imply the strict digital contraction. So let $x, y \in X$ be such that $x \neq y$. Then $d(x, y) > 0$. Let $\epsilon = d(x, y) > 0$. Then by the condition of weakly uniformly strict digital contraction, there exists a $\delta > 0$ such that $\epsilon \leq d(x, y) < \epsilon + \delta$ implies $d(Tx, Ty) < \epsilon$ that is $d(Tx, Ty) < d(x, y)$. We now prove that T is (κ, κ)-continuous. Let $a \in X$ and let $\epsilon > 0$. Let $\delta = \epsilon$. Then if $d(a, b) < \delta$, we have $d(Ta, Tb) < d(a, b) < \delta = \epsilon$. Thus given $\epsilon > 0$, there exists a $\delta > 0$ such that $d(a, b) < \delta$ implies $d(Ta, Tb) < \epsilon$. Hence the mapping T is (κ, κ)-continuous. Next we show that if a fixed point of T exists then it is unique. Let $a, b \in X$ be fixed points of T. That is $Ta = a$ and $Tb = b$. Then we see by condition of strict digital contraction that, if $a \neq b$, then $d(Ta, Tb) = d(a, b) < d(a, b)$. Thus $d(a, b) = 0$ and hence $a = b$. Next we proceed to show that the sequence $\{x_n\}_{n=1}^{\infty} = \{T^n x\}_{n=1}^{\infty}$ is a Cauchy sequence for every $x \in X$. Consider the sequence $\{u_n\}_{n=1}^{\infty} = \{d(x_n, x_{n+1})\}_{n=1}^{\infty}$. Since T satisfy the condition of strict digital contraction, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(T^n x, T^{n+1} x) \\ &= d(T(T^{n-1}x), T(T^n x)) \\ &< d(T^{n-1}x, T^n x) \\ &= d(x_{n-1}, x_n) \\ \therefore d(x_n, x_{n+1}) &< d(x_{n-1}, x_n) \end{aligned}$$

Thus the sequence $\{u_n\}_{n=1}^{\infty} = \{d(x_n, x_{n+1})\}_{n=1}^{\infty}$ is decreasing sequence. It is also bounded below (by 0). Hence it is a convergent sequence. Let $\lim_{n\to\infty} u_n = L$. If $L > 0$ then letting $\epsilon = L > 0$, by the condition of weakly uniformly strict digital contraction, there exists a $\delta > 0$, such that $L \leq d(x_n, x_{n+1}) < L + \delta$ implies $d(x_{n+1}, x_{n+2}) < L$. Then for all $m > n + 1, n + 2$, we have $d(x_m, x_{m+1}) < L$ (since the sequence $\{u_n\}_{n=1}^{\infty}$ is decreasing sequence). But then $\lim_{n\to\infty} u_n < L$. This is a contradiction. Therefore $\lim_{n\to\infty} d(x_n, x_{n+1}) = L = 0$. Now we prove that the sequence $\{x_n\}_{n=1}^{\infty} = \{T^n x\}_{n=1}^{\infty}$ is a Cauchy sequence for all $x \in X$. This we show by contradiction method. So let us assume that $\{x_n\}_{n=1}^{\infty} = \{T^n x\}_{n=1}^{\infty}$ is not a Cauchy sequence for some $x \in X$. Then there exists $2\epsilon > 0$ such that

$$\limsup_{n\to\infty} d(x_m, x_n) > 2\epsilon$$

By hypothesis, there exists $\delta > 0$ such that $\epsilon \leq d(x, y) < \epsilon+\delta$ implies $d(Tx, Ty) < \epsilon$. This condition is true even if we replace δ by $\Delta = \min(\delta, \epsilon)$. Since

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0,$$

we can find M such that $d(x_M, x_{M+1}) < \frac{\Delta}{3}$. Choose $m, n > M$ so that $d(x_m, x_n) > 2\epsilon$. For $m \leq j \leq n$, we have

$$|d(x_m, x_j) - d(x_m, x_{j+1})| \leq d(x_j, x_{j+1}) < \frac{\Delta}{3}$$

This implies that there exists $m \leq j \leq n$ with

$$\epsilon + \frac{2\Delta}{3} < d(x_m, x_j) < \epsilon + \Delta$$

However, for all m and j,

$$\begin{aligned} d(x_m, x_j) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{j+1}) + d(x_{j+1}, x_j) \\ \therefore d(x_m, x_j) &\leq d(x_m, x_{m+1}) + \epsilon + d(x_j, x_{j+1}) \\ &< \frac{\Delta}{3} + \epsilon + \frac{\Delta}{3} \\ &= \epsilon + \frac{2\Delta}{3} \end{aligned}$$

This is a contradiction to the fact that $\epsilon + \frac{2\Delta}{3} < d(x_m, x_j) < \epsilon + \Delta$. Hence $\{x_n\}_{n=1}^{\infty} = \{T^n x\}_{n=1}^{\infty}$ must be Cauchy sequence for all $x \in X$. Since (X, d, κ) is a complete digital metric space, there exists a point z_x such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n x = z_x$$

for all $x \in X$. Since T is (κ, κ)-continuous, we have

$$Tz_x = T\big(\lim_{n\to\infty} T^n x\big) = \lim_{n\to\infty} T^{n+1} x = z_x.$$

Thus z_x is a fixed point of T. As we have already observed that the fixed point is unique, we conclude that all the z_x are same. Hence the theorem is proved □

4 An Application of Fixed Point Theorems to Digital Images

In this section we discuss the application of fixed point theorems that we have proved in the last section. We first recall that the theorem (3.1) uses the function $\psi : (0, \infty) \to (0, \infty)$, which is monotone non-decreasing and satisfy $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$. Note that function $\psi(t)$ may be some non-linear function.

Fractal geometry has attracted the attention of many Mathematicians. Mandelbrot first coined the term fractal. Roughly a fractal is a geometric shape, every part of which is a reduced copy of the whole. The following are examples of fractals.

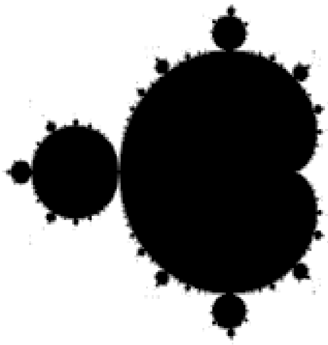

Fig. 4. Madelbrot Set

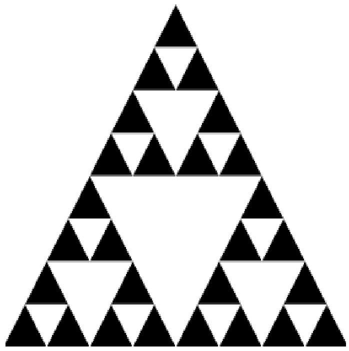

Fig. 5. Sierpinski Triangle

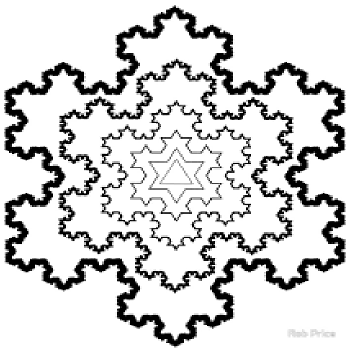

Fig. 6. Koch Snowflake

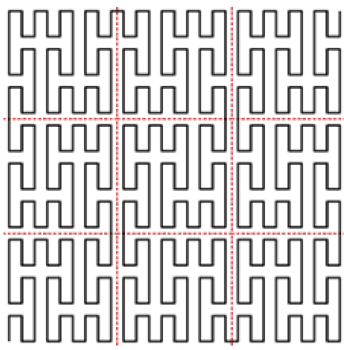

Fig. 7.Peano Curve

Many real world objects such as coastlines, mountains, trees, clouds are approximated by using fractals. Mandelbrot's book The Fractal Geometry of Nature [16] attracted a wide range of attention.

Hutchinson [17] initiated the theory of iterated function system (IFS). Barnsley first recognized the potential of the fractals for the image compression and applied the theory of IFS. Barnsley published his book Fractals Everywhere [18]. He also published a paper on fractal image compression [19]. This research activity attracted many researchers in applied mathematics and computers towards the fractals.

The original image is segmented into parts such that each part is nearly same as a reduced copy of the original image. The union of all the segments is then close enough to the original image. Thus the images with global self similarity are encoded with extreme efficiency [20, 21, 22]. Unfortunately, a general image is not always globally self similar. In such images, self similarity exists only locally amongst different small parts of it. See the following image.

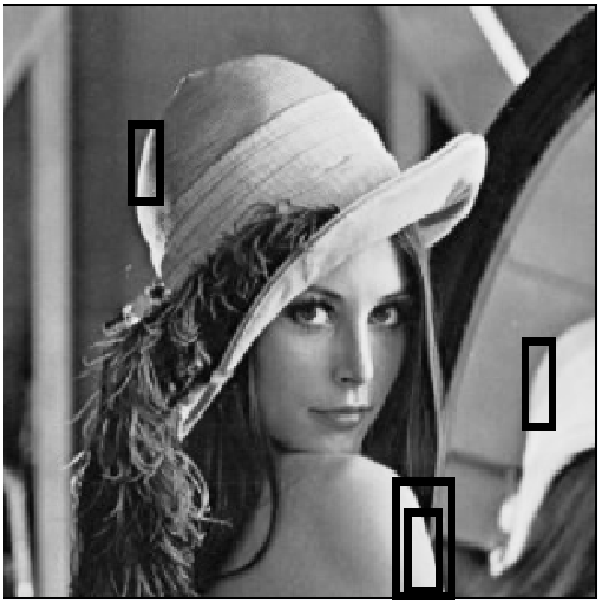

Figure 8: Self Similarity in the Image of Lenna

It has been observed that all the images in nature contain a considerable amount of affine redundancy. The affine redundancy means, large segments of the image look like the small segments of the same image. Large segments are known as domain blocks whereas small segments as range blocks. We can find an affine transformation (a combination of rotation, reflection, scaling and shifting transformation) that transforms a domain block to the suitable range block. The parameters of the transformation constitutes a fractal code. Thus a range block is approximated by applying an affine transformation on suitably chosen domain block. Since the mappings reduces the size of the domain block, it is a contractive mapping. Fractal image compression works as follows:

1. The image is partitioned into non-overlapping range blocks. Generally, the partition of an image may have any arbitrary shape (square, rectangles, triangles, quadrilaterals or any polygon [23, 24, 25]).

2. The same image is partitioned into overlapping domain blocks. Domain blocks are larger in size than the range blocks in order to maintain contractive condition.

3. Finally the image is encoded by using a suitable affine transformation which maps a domain block to a best fitted range block.

4. To achieve the decompression, exactly opposite is done. Inverse affine transform is applied to recover the image. Usually 8 to 9 inverse iterations are applied on the encoded image to decode the image. The iteration starts with any arbitrary image. Successive application of the affine map gives the sequence of images that ultimately converge to a fixed image (by fixed point theorem of Banach).

Thus the contractive mappings and fixed point theorem is at the core of the fractal image compression. Important aspect of the fractal image decoding is resolution independence. That means we may compress a 128×128 image, and decompress it to any size, say 64×64 or 256×256. Fractal image compression produces better reconstructed image than that of JPEG (Joint Photographic Expert Group) technique. This we can see from the difference between images 17 and 18 in the following figure. Original file size is 1036 bytes and it is compressed to an image of size is 996 bytes in 5 seconds.

Fig. 9.Original Image

Fig. 10. Compressed Image

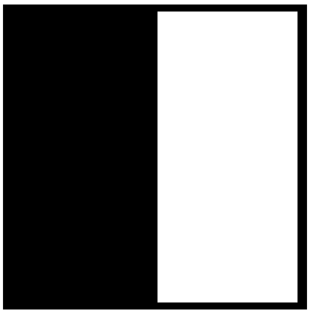

Fig. 11. Iteration 1 (start with any image)

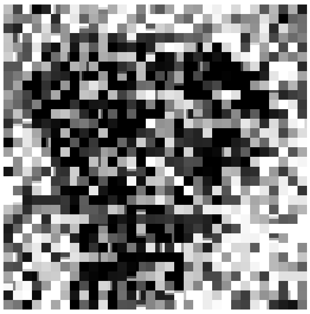

Fig. 12.Iteration 2

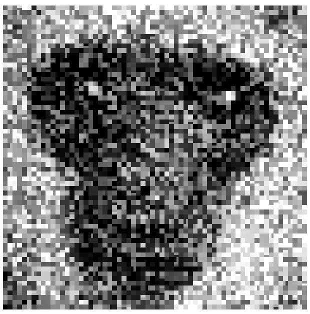

Fig. 13. Iteration 3

Fig. 14.Iteration 4

Fig. 15.Iteration 5

Fig. 16.Iteration 6

Fig. 17.Iteration 7

Fig. 18.JPEG Image

Outstanding results of fractal image compression inspired researchers to apply this technique to video encoding[26]. Although it is true that JPEG is the most widely used technique for the image compression, there is no option but to use fractal image compression at least where very high compression ratio is required. But it has a drawback of time consuming encoding. Fractal decoding of images is based on the fixed point theorems [27]. We see the mathematical modeling in the fractal image compression in the remaining part of the article.

Definition 4.1. Let (X, d, κ) be a digital metric space. $T_i : X \to X,\ i = 1, 2, \cdots, n$ be the contractions with the contraction factors $\lambda_i,\ i = 1, 2, \cdots, n$. Let $\lambda = \max\{\lambda_i,\ i = 1, 2, \cdots, n\}$. Then the set of contractions $\{T_i,\ i = 1, 2, \cdots, n\}$ with the contraction factor λ is called an iteration function system (IFS).

First we discuss fractal image compression for binary images.

Definition 4.2. For a digital metric space $(\mathbb{R}^2, d, \kappa)$ affine transformations are defined as the IFS $\{T_i,\ i = 1, 2, \cdots, n\}$, where

$$T_i \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p_i & q_i \\ r_i & s_i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_i \\ u_i \end{bmatrix}$$

Here

$$\begin{bmatrix} p_i & q_i \\ r_i & s_i \end{bmatrix}$$

amounts to scaling and rotation, and

$$\begin{bmatrix} t_i \\ u_i \end{bmatrix}$$

represents the translation. We observe that $T_i,\ i = 1, 2, \cdots$ are contractions if and only if

$$\begin{vmatrix} p_i & q_i \\ r_i & s_i \end{vmatrix} < 1$$

Let $\mathcal{P}(X)$ be the set of all non-empty compact subsets of the digital metric space (X, d, κ).

Definition 4.3. The Hausdorff distance between the two sets A, B in $\mathcal{P}(X)$ is defined by

$$H(A, B) = \max\Big\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\Big\}$$

where $d(x, B) = \inf_{y\in B} d(x, y)$ and $d(y, A) = \inf_{x\in A} d(y, x)$.

Let $\{T_i,\ i = 1, 2, \cdots, n\}$ be the IFS defined on the digital metric space (X, d, κ) with the contractive factor λ. Define $T : \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$TA = \bigcup_{i=1}^{n} T_i A \tag{4.1}$$

for all $A \in \mathcal{P}(X)$. Then T is a contraction on $\mathcal{P}(X)$ with respect to the Hausdorff matrix, that is

$$H(TA, TB) \le \lambda H(A, B)$$

for all $A, B \in \mathcal{P}(X)$.

Remark 4.4. The union defined above in equation (4.1) is the assembly of the reduced copies of the whole image.

The following theorem is the key result for the application in fractal image compression.

Theorem 4.5. [27] Consider the matrix space $(\mathcal{P}(X), H)$ and the mapping $T : \mathcal{P}(X) \to \mathcal{P}(X)$ defined as in (4.1). Then the mapping T has a unique fixed point Z in $\mathcal{P}(X)$. Moreover, $\lim\limits_{n\to\infty} T^n A = Z$ for any $A \in \mathcal{P}(X)$.

Remark 4.6. The equation $\lim\limits_{n\to\infty} T^n A = Z$ in the theorem 4.5 suggests that we can start with any random block in the domain and successively apply the affine transformation on it to get the output image.

Now we consider the inverse problem. Suppose the image Z, that is to be compressed, is the fixed point of the affine transformation T. Then the aim is to construct the IFS so that, we get $\lim\limits_{n\to\infty} T^n A = Z$.

We refer [20] to see that Barnsley theorem is useful in the inverse problem. Actually, the following theorem, called collage theorem addresses the problem.

Theorem 4.7. [27] Let (X, d, κ) be a digital metric space and let $A \in \mathcal{P}(X)$. Given any $\epsilon \ge 0$. Select an IFS $\{T_i,\ i = 1, 2, \cdots, n\}$ with the contraction factor $0 \le \lambda < 1$, such that

$$H\Big(A, \bigcup_{i=1}^{n} T_i A\Big) \le \epsilon$$

Then

$$H(A, Z) \leq \frac{\epsilon}{1-\lambda}$$

where $Z = \lim_{n\to\infty} T^n A$.

Practically the collage theorem is applied as follows: Given any $A \in \mathcal{P}(X)$, draw an outline of A. Cover it by ϵ close smaller copies of the outline. These copies are called collage. There is a unique affine map T_i from the outline of A onto each of the collage. The collage theorem 4.7 states the that more accurately the image A is covered by the collage, the more close is the image Z to the image A.

Now let us consider the case of grayscale images. A greyscale image is one in which the value of each pixel is a single sample representing only an amount of light, that is, it carries only intensity information. Images of this sort, also known as black-and-white or monochrome. These images are composed exclusively of shades of gray, varying from black at the weakest intensity to white at the strongest. The set X of all the grayscale image can be represented by a real valued function $z = f : S^2 \to [a, b]$, where $S^2 \subset \mathbb{R}^2$ is the support of the image,$[a, b] \subset \mathbb{R}$ is an interval representing the gray levels in the image. This space becomes a complete metric space with the supremum metric [28],

$$d(f, g) = \sup |f - g|$$

and L_2metric

$$d(f, g) = \left\{ \int_{S^2} [f(x, y) - g(x, y)]^2 dxdy \right\}^{\frac{1}{2}}$$

for all $f, g \in X$.

Both fixed point theorem and collage theorem hold true for this space.

Theorem 4.8. [27] Let $T : X \to X$ be a contraction defined on the complete metric space (X, d). Then there exists a unique fixed point $Z \in X$. Moreover, $Z = \lim_{n\to\infty} T^n f$ for any $f \in X$.

Let f_0 be the original image to be compressed. The aim is to find the IFS so that $f_0 = \lim_{n\to\infty} T^n f$. The collage theorem resolves this issue.

Theorem 4.9. [27] Let $T : X \to X$ be a contraction with contraction factor λ. Then

$$d(f_o, f) \leq \frac{1}{1-\lambda} d(f_0, T f_o)$$

As the natural images are rarely self similar, the practical fractal coding is a block coding. Divide the image into non-overlapping range blocks $R_1, R_2, \cdots, R_n$. Then divide the image into overlapping domain blocks $D_1, D_2, \cdots, D_m$. Domain blocks are larger in size to make the mapping contractive. Usually the domain block is chosen double the range block. The contraction is defined block wise as

$$Tf = \bigcup_{i=1}^{n} T_i f$$

where $T_i : f|_{D_i} \to f|_{R_i}$ is a contraction. We have

$$Tf = \bigcup_{i=1}^{n} Tf|_{R_i} = \bigcup_{i=1}^{n} T_i f|_{D_i}$$

Thus we see that fractal code is constructed by defining the IFS separately for each range block R_i. Encoding procedure is to find IFS from domain D_i to the range R_i, so that the domain block is close to the original range block.
Observe that T_i is effectively three dimensional transformation since
1. It spatially shrinks the domain to range
2. It contracts the intensity of the domain block.
Let

$$T_i(x, y, z) = (G_i(x, y), I_i(z)) : D_i \times I \to R_i \times I,$$

where G_i is geometric part and I_i is an intensity part. Now the geometric part G_i is contracting and rotating the domain block so as to fit to the range block. So it is a composition of two mappings, say, $h_i : D_i \to R_i$, a contraction and $\theta_i : R_i \to R_i$, a rotation. Thus the geometric part is $G_i = \theta_i \circ h_i$. The matrix form of G_i becomes

$$G_i \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p_i & q_i \\ r_i & s_i \end{bmatrix} \begin{bmatrix} x - D_{x_i} \\ y - D_{y_i} \end{bmatrix} + \begin{bmatrix} R_{x_i} \\ R_{y_i} \end{bmatrix}$$

The domain D_i transformed to the range block R_i under the geometric transformation not only shrinks its shape but also increases its contrast. Let us denote the contrast or intensity of grayscale caused by the geometric transformation G_i as $g_i(f) : [a, b] \to [a, b]$. The pixel intensity is taken as the local average of those pixels from the domain block before the geometric mapping. Let the pixel intensity be denoted by u. Thus

$$u(x, y) = g_i f|_{D_i} = g(f(G_i^{-1}(x, y)))$$

for all $(x, y) \in R_i$. Further intensity mappings are now performed according to

$$m_i f : [a, b] \to [a, b]$$

where m_i consists of scaling and intensity shifting. Let

$$v : m_i(u) = \lambda_i u(x, y) + t_i$$

The total intensity transformation becomes

$$I_i(f) = m_i \circ g_i(f) : [a, b] \to [a, b]$$

Note again that both geometric and intensity mappings are contractions, so that we can use the fixed point theorem. The IFS for the point (x, y) and intensity z in the domain block D_i is defined as

$$T_i \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_i & q_i & 0 \\ r_i & s_i & 0 \\ 0 & 0 & \lambda_i \end{bmatrix} \begin{bmatrix} x - D_{x_i} \\ y - D_{y_i} \\ g_i(z) \end{bmatrix} + \begin{bmatrix} R_{x_i} \\ R_{y_i} \\ t_i \end{bmatrix}$$

where $(p_i, q_i, r_i, s_i, D_{x_i}, D_{y_i}, R_{x_i}, R_{y_i})$ amounts to the geometric part and (λ_i, t_i, g_i) represents the intensity part. We choose the parameters s_i and t_i so that the the contraction map T_i takes a domain block as close to range block as possible. In particular let

$$E = \frac{1}{MN} \sum\sum_{(x,y) \in R_i} [\lambda_i u + t_i - f]^2$$

where $M \times N$ is the size of the range block. From the calculus of several variables, the minimum value of E is attained where the partial derivatives of E with respect to λ_i and t_i become zero. Finding these partial derivatives and equating them to zero we get two equations. Solving these linear equations for λ_i and t_i, we get

$$\lambda_i = \frac{MN \sum\sum_{(x,y)\in R_i} fu - \Big(\sum\sum_{(x,y)\in R_i} f\Big)\Big(\sum\sum_{(x,y)\in R_i} u\Big)}{MN \sum\sum_{(x,y)\in R_i} u^2 - \Big(\sum\sum_{(x,y)\in R_i} u\Big)^2}$$

$$t_i = \frac{1}{MN}\Big(\sum\sum_{(x,y)\in R_i} f - \lambda_i \sum\sum_{(x,y)\in R_i} u\Big)$$

The IFS is described by the following five numbers:

1. θ_i - The index of symmetries,
2. D_{x_i} - The x coordinate of the domain block D_i,
3. D_{y_i} - The y coordinate of the domain block D_i,
4. λ_i - The contraction factor,
5. t_i - The shift in the gray level.

Since each of the mapping is contractive, we can apply the fixed point theorem for decoding. Start with any image f and apply T successively to compute $Tf,\ T^2f, \cdots$ until the sequence converges. Normally we need to apply T 8 to 10 times.

Having considered some basic background of fractal image compression, and its technique in practical applications, now we turn our attention to application of our fixed point theorem. As we know, the researchers work towards either maximizing quality of the image at a fixed transmission rate or minimize the transmission time for a given quality of the image. As an application, we try to improve the image quality at a given rate of transmission. To achieve this goal, we replace the constant λ_i in the contraction mapping by a non-linear contractive function.

Definition 4.10. Let T be a mapping on the metric space (X, d). If there exists function $0 \leq |\lambda(x, y)| < 1$ such that

$$d(Tf, Tg) \leq \lambda(x, y)d(f, g)$$

for all $f, g \in X$, then the mapping T is said to be a contraction mapping and the function $\lambda(x, y)$ is called a contractive function.

Example 4.11. Consider the function $\psi : (0, \infty) \to (0, \infty)$ defined by

$$\psi(t) = \begin{cases} \frac{te^t}{(n+1)(1+e^t)} & \text{if, } \frac{1}{n+1} < t \leq \frac{1}{n}, \\ 0 & \text{if, } t = 0, \\ 1 & \text{if, } t > 1 \end{cases} \tag{4.2}$$

The function satisfy the following:

1. $\psi(t)$ non-decreasing on $[0, \infty)$,
2. $\lim\limits_{n\to\infty} \psi^n(t) = 0$,
3. $\psi(t)$ is a non-linear function.

Let (X, d) be a metric space and suppose that $T : X \to X$ satisfy $d(Tx, Ty) \leq \psi(d(x, y))$ where ψ is as defined in (4.2). Then we get,

$$\begin{aligned} d(Tx, Ty) &\leq \psi(d(x, y)) \\ &= d(x, y)\frac{e^{d(x,y)}}{(n+1)(1+e^{d(x,y)})} \\ &= \lambda(x, y)d(x, y) \end{aligned}$$

where

$$\lambda(x, y) = \frac{e^{d(x,y)}}{(n+1)(1+e^{d(x+y)})}$$

Here

$$|\lambda(x, y)| = \left|\frac{e^{d(x,y)}}{(n+1)(1+e^{d(x+y)})}\right| < 1$$

Hence $\lambda(x, y)$ is a contractive function. By the theorem 3.1, T has a unique fixed point in X.

Now we consider the advantage of using a non-linear contractive functions. A cluster of planes which are not parallel to the Z axis has the general form $z = ax + by + c$. Then we construct the contractive functions by limiting the dynamic range of these planes to $(-1, 1)$ with the following mapping

$$\lambda(x, y, a, b, c) = \pm\frac{1}{e^{ax+by+c}}$$

a, b, c to be optimized. $\lambda(x, y)$ takes positive values when the angle between a domain normal direction and a range normal direction is less than $\frac{\pi}{2}$, and negative values, if the angle is larger than $\frac{\pi}{2}$. The following figure shows a domain block that is mapped into a steeper range block under a nonlinear contractive function.

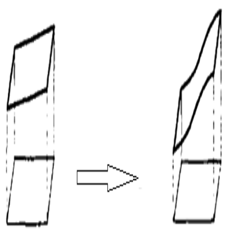

Fig. 19. Transformation under non linear contractive function

The optimization objective function can be written as

$$E = \frac{1}{MN}\sum_{(x,y)\in R_i}\sum[\lambda_i(x, y, p_i, q_i, r_i,)u(x, y) + t_i - f(x, y)]^2$$

The peak signal-to-noise ratio (PSNR) is used to measure the difference between two images. The PSNR is measured in decibels and defined as follows:

$$PSNR = 10\log_{10}\left\{\frac{B^2}{Mean\ Squared\ Error}\right\}$$

Here B is the largest signal amplitude. For the image of size $N \times N$ the mean squared error is given by

$$MSE = \frac{1}{N^2}\sum_{x=1}^{N}\sum_{y=1}^{N}[f(x,y) - \hat{f}(x,y)]^2$$

where f and $\hat{f}$ are original and reconstructed images respectively. As per the experiments conducted for the nonlinear contractive function method and traditional method of fractal image compression, we get the following table of observation. It is clear that the non linear fractal coding technique gives a better reconstructed image.

Table 1. PSNR calculated for non linear contractive method and L_2 metric method

Number of Iterations	1	2	3	4	5	6	7	8	9	10
PSNR (Nonlinear)	19.91	25.04	29.19	30.66	31.01	31.05	31.05	31.05	31.05	31.05
PSNR (L_2 metric)	18.03	21.00	23.66	26.64	29.13	30.25	30.45	30.46	30.45	30.35

5 Conclusion

Purpose of this paper is to give a digital version of some important generalizations of Banach contraction mapping principle. We have extended the Banach contraction mapping principle to digital images by using a non-decreasing function. Also weakly uniformly strict contraction is incorporated to establish another fixed point theorem. An attempt has been made to give an application of our fixed point theorems to fractal image compression and improve the perceptual quality of transformed digital image.

Acknowledgements

Authors are thankful to the anonymous referees for their critical remarks. Their valuable suggestions is a key to improved the article, especially the last section.

Competing Interests

Authors have declared that no competing interests exist.

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